A Class of Energetically Rigid Gravitational Fields

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In Einstein's theory, the physics of gravitational fields is reflected by the geometry of the curved space-time manifold. One of the methods for a study of the geometrical properties of space-time consists in regarding it, locally, as embedded in a higher-dimensional flat space. In this paper, metrics admitting a 3-parameter group of motion are considered which form a generalization of spherically symmetric gravitational fields. A subclass of such metrics can be embedded into a five-dimensional flat space. It is shown that the second fundamental form governing the embedding can be expressed entirely by the energy-momentum tensor of matter and the cosmological constant. Such gravitational fields are called energetically rigid. As an application gravitating perfect fluids are discussed.

1. Introduction

In Einstein's theory of gravitation the investigation of the physics of gravitational fields amounts to a study of the geometrical properties of curved space-time V_4 whose metric tensor $g_{\alpha\beta}$ satisfies the field equations with cosmological constant Λ [1]:

$$G_{\alpha\beta} + \Lambda\, g_{\alpha\beta} = -\varkappa\, T_{\alpha\beta}\,. \tag{1.1}$$

One of the methods for such a study consists in regarding V_4, locally, as a subspace of an auxiliary flat space E_N of $N \leqq 10$ dimensions. The number $p = N - 4 \leqq 6$ is called the (embedding) class. It is an arithmetic invariant of space-time. If y^{A} ($A = 1, 2, \ldots, N$) are the coordinates of the single chart covering E_N and x^a ($a = 0, 1, 2, 3$) coordinates of V_4 then the local and isometric embedding of V_4 into E_N is described by

$$y^{A} = y^{A}(x^a) \tag{1.2 a}$$

$$g_{\alpha\beta} = \eta_{AB}\, y^{A}{}_{,\,\alpha}\, y^{B}{}_{,\,\beta}\,. \tag{1.2 b}$$

In Eq. (1.2 a) $g_{\alpha\beta}$ and $\eta_{AB} := e_A\, \delta_{AB}$ with $e_A{}^2 = 1$ are the components of the metric tensors of curved space-time V_4 and flat E_N, respectively.

From Eqs. (1.2 a, b) by covariant differentiation, a system of algebraic and differential equations, the Gauß-Codazzi-Ricci equations, can be derived [2]. In the case of class one this set of equations simplifies considerably; the remaining equations are

$$R_{\alpha\beta\gamma\delta} = e\,(b_{\alpha\gamma}\, b_{\beta\delta} - b_{\alpha\delta}\, b_{\beta\gamma}) \tag{1.3 a}$$

$$b_{\alpha\beta;\,\gamma} - b_{\alpha\gamma;\,\beta} = 0\,. \tag{1.3 b}$$

The covariant, symmetric tensor field of rank 2 $b_{\alpha\beta}$ is called second fundamental form of space-time. The relationship between the Riemannian curvature tensor $R_{\alpha\beta\gamma\delta}$ and $b_{\alpha\beta}$ cannot be expected to be unique (possibility of bending subspaces).

The tensor field $b_{\alpha\beta}$ on V_4 not being determined in terms of $g_{\alpha\beta}$ and its derivatives alone, it eludes physical interpretation in general. For such an interpretation, the case is most interesting in which the second fundamental form $b_{\alpha\beta}$ is determined (up to sign) by $g_{\alpha\beta}$ (and its derivatives). Such a space-time will be called intrinsically rigid. As $b_{\alpha\beta}$ is a tensor it must depend on $R_{\alpha\beta\gamma\delta}$. A space-time V_4 of class one whose second fundamental form is determined entirely by the Einstein tensor $G_{\alpha\beta}$ is termed energetically rigid. According to Eq. (1.1) $b_{\alpha\beta}$ then can be expressed by the energy-momentum tensor $T_{\alpha\beta}$ of matter and the cosmological constant Λ.

In Sect. 2 gravitational fields admitting a 3-parameter group of motion are considered which form a generalization of spherically symmetric metrics. In general, they are of class $p \leqq 3$. In Sect. 3, only metrics with class $p = 1$ are kept and shown to be energetically rigid. It seems rather interesting that for the large class of solutions of Einstein's field equations considered the embedding is fully determined by the material sources. Explicit expressions for $b_{\alpha\beta}(G_\lambda{}^\varkappa)$ are given and applied to the case of gravitating perfect fluids in Section 4.

2. Metrics with Isometry Group G_3 (2, s)

In the following metrics are considered which admit a 3-parameter group of isometries with two-dimensional, spacelike orbits. The group being maximal on the orbits, these are spaces of constant

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curvature. For details, see Goenner and Stachel [3]. This family of metrics is a generalization of spherically symmetric manifolds and contains many well-known exact solutions of Einstein's equations, for example the Schwarzschild-, De-Sitter- and Taub's plane symmetry metric, the degenerated vacuum fields $A_1 - A_3$ of Ehlers and Kundt [4]. The manifolds with $G_3(2, s)$ are contained among metrics with local isotropy studied by Cahen and Defrise [5].
The metric may be put into the form:

$$\mathrm{d}s^2 = e^{2\nu}(\mathrm{d}x^0)^2 - e^{2\lambda}(\mathrm{d}x^1)^2 - e^{2\mu}\,\mathrm{d}\omega^2 \tag{2.1}$$

where

$$\mathrm{d}\omega^2 = (\mathrm{d}x^2)^2 + \Sigma^2(\mathrm{d}x^3)^2 \tag{2.2}$$

is the metric on the group orbits and

$$\Sigma(x^2) = \begin{Bmatrix} \sin x^2 \\ \sinh x^2 \\ 1 \end{Bmatrix} \text{ if the (constant) curvature of the orbit is } \begin{Bmatrix} >0 \\ <0 \\ =0 \end{Bmatrix}. \tag{2.3}$$

Furthermore,

$$\nu = \nu(x^0, x^1), \quad \lambda = \lambda(x^0, x^1), \quad \mu = \mu(x^0, x^1).$$

The embedding class p of the familiy (2.1) is $p \leqq 3$; if the orbits are not flat $p \leqq 2$.

The non-vanishing components of the curvature tensor for (2.1) are

$$\begin{aligned}
R_{01}{}^{01} &= -e^{-2\nu}(\ddot\lambda + \dot\lambda^2 - \dot\lambda\,\dot\nu) + e^{-2\lambda}(\nu'' + \nu'^2 + \nu'\,\lambda') \\
R_{02}{}^{02} &= R_{03}{}^{03} = -e^{-2\nu}(\ddot\mu + \dot\mu^2 - \dot\mu\,\dot\nu) + \mu'\,\nu'\,e^{-2\lambda} \\
R_{12}{}^{12} &= R_{13}{}^{13} = e^{-2\lambda}(\mu'' + \mu'^2 - \lambda'\,\mu') - \dot\mu\,\dot\lambda\,e^{-2\nu} \\
R_{12}{}^{02} &= R_{13}{}^{03} = e^{-2\nu}(-\dot\mu' - \dot\mu\,\mu' + \dot\lambda\,\mu' + \nu'\,\dot\mu) \\
R_{23}{}^{23} &= e^{-2\mu}\,\Sigma^{-1}\,\Sigma_{,22} - \dot\mu^2\,e^{-2\nu} + \mu'^2\,e^{-2\lambda}
\end{aligned} \tag{2.4}$$

where $\dot\mu := \mathrm{d}\mu/\mathrm{d}x^0$, $\mu' := \mathrm{d}\mu/\mathrm{d}x^1$. A calculation of the Einstein tensor leads to four independent non-vanishing components:

$$\begin{aligned}
G_0{}^0 &= e^{-2\mu}\,\Sigma^{-1}\,\Sigma_{,22} - e^{-2\nu}(\dot\mu^2 + 2\,\dot\mu\,\dot\lambda) \\
&\quad + e^{-2\lambda}(3\,\mu'^2 + 2\,\mu'' - 2\,\lambda'\,\mu'), \\
G_1{}^1 &= e^{-2\mu}\,\Sigma^{-1}\,\Sigma_{,22} - e^{-2\nu}(3\,\dot\mu^2 + 2\,\ddot\mu - 2\,\dot\mu\,\dot\nu) \\
&\quad + e^{-2\lambda}(\mu'^2 + 2\,\mu'\,\nu'), \\
G_1{}^0 &= 2\,e^{-2\nu}(\dot\mu' + \dot\mu\,\mu' - \dot\lambda\,\mu' - \nu'\,\dot\mu), \\
G_2{}^2 &= G_3{}^3 = e^{-2\nu}(-\ddot\lambda - \dot\lambda^2 + \dot\lambda\,\dot\nu - \ddot\mu - \dot\mu^2 - \dot\lambda\,\dot\mu + \dot\mu\,\dot\nu) \\
&\quad + e^{-2\lambda}(\nu'' + \nu'^2 + \nu'\,\lambda' \\
&\quad + \mu'' + \mu'^2 + \mu'\,\nu' - \lambda'\,\mu'),
\end{aligned} \tag{2.5}$$

whence, together with Eq. (2.4), follows:

$$\begin{aligned}
R_{01}{}^{01} &= G_2{}^2 - \tfrac12(G_0{}^0 + G_1{}^1) + R_{23}{}^{23}, \\
R_{02}{}^{02} &= R_{03}{}^{03} = \tfrac12(G_1{}^1 - R_{23}{}^{23}), \\
R_{12}{}^{12} &= R_{13}{}^{13} = \tfrac12(G_0{}^0 - R_{23}{}^{23}), \\
R_{12}{}^{02} &= R_{13}{}^{03} = -\tfrac12 G_1{}^0, \\
R_{02}{}^{12} &= R_{03}{}^{13} = -\tfrac12 G_0{}^1.
\end{aligned} \tag{2.6}$$

3. Restriction to Class One

Now, metrics (2.1) of class one are studied *. If rank $b_{\alpha\beta} \geqq 3$, $(\mathcal{L}_\eta)\,b_{\alpha\beta} = 0$ is a consequence of $(\mathcal{L}_\eta)\,g_{\alpha\beta} = 0$. $\mathcal{L}_\eta$ denotes the Lie-derivative, c. f. Yano [6]. Thus, the second fundamental form necessarily has the form

$$\begin{aligned}
b_{\alpha\beta} &= b_{00}\,\delta_\alpha{}^0\,\delta_\beta{}^0 + b_{01}\,\delta_{(\alpha}{}^0\,\delta_{\beta)}{}^1 + b_{11}\,\delta_\alpha{}^1\,\delta_\beta{}^1 \\
&\quad + b_{22}(\delta_\alpha{}^2\,\delta_\beta{}^2 + \Sigma^2\,\delta_\alpha{}^3\,\delta_\beta{}^3).
\end{aligned} \tag{3.1}$$

After insertion of Eq. (3.1) into the Gauss Eq. (1.3 a), the following relation is found to hold:

$$R_{23}{}^{23}\,R_{01}{}^{01} + R_{12}{}^{02}\,R_{03}{}^{13} = R_{02}{}^{02}\,R_{13}{}^{13}. \tag{3.2}$$

For spherically symmetric gravitational fields Eq. (3.2) was derived already by Eiesland [7].
Now, if $R_{23}{}^{23} \neq 0$,

$$\begin{aligned}
b_0{}^0 &= -\varepsilon_1\,R_{02}{}^{02}\,(e\,R_{23}{}^{23})^{-1/2}, \\
b_1{}^1 &= -\varepsilon_1\,R_{12}{}^{12}\,(e\,R_{23}{}^{23})^{-1/2}, \\
b_1{}^0 &= -\varepsilon_1\,R_{12}{}^{02}\,(e\,R_{23}{}^{23})^{-1/2}, \\
b_2{}^2 &= b_3{}^3 = -e\,\varepsilon_1\,(e\,R_{23}{}^{23})^{+1/2}.
\end{aligned} \tag{3.3}$$

In Eq. (3.3) e and ε_1 are sign-factors ($e^2 = \varepsilon_1{}^2 = 1$) with e coming from the Gauss equation. By solving Eq. (3.2) for $R_{23}{}^{23}$ and using Eq. (2.6) I obtain the following expressions for $b_\beta{}^\alpha$:

$$\begin{aligned}
b_0{}^0 &= -\varepsilon_1\,\Omega^{-1/2}\left[\tfrac{5}{12}G_1{}^1 - \tfrac{1}{12}G_0{}^0 + \tfrac13 G_2{}^2 \mp \tfrac{1}{12}\Delta^{1/2}\right], \\
b_1{}^1 &= -\varepsilon_1\,\Omega^{-1/2}\left[\tfrac{5}{12}G_0{}^0 - \tfrac{1}{12}G_1{}^1 + \tfrac13 G_2{}^2 \mp \tfrac{1}{12}\Delta^{1/2}\right], \\
b_1{}^0 &= \tfrac12\,\varepsilon_1\,\Omega^{-1/2}\,G_1{}^0, \\
b_2{}^2 &= b_3{}^3 = -e\,\varepsilon_1\,\Omega^{1/2}
\end{aligned} \tag{3.4}$$

with

$$e\,\Omega := \tfrac16(G_0{}^0 + G_1{}^1) - \tfrac23 G_2{}^2 \pm \tfrac16\Delta^{1/2} \tag{3.4 a}$$

and

$$\Delta := (4\,G_2{}^2 - G_0{}^0 - G_1{}^1)^2 + 12\,G_0{}^0\,G_1{}^1 - 12\,G_1{}^0\,G_0{}^1. \tag{3.4 b}$$

Equation (3.3) shows that manifolds with $G_3(2, s)$ of class one are intrinsically rigid while, from Eq.

* Among them, for ex., the Friedman cosmological models can be found.

(3.4), it is seen that such manifolds are even energetically rigid. There is no need to investigate the Codazzi Eqs. (1.3 b) if rank $b_{\alpha\beta} \geqq 4$. According to Thomas [8] in this case Eq. (1.3 b) follows from (1.3 a).

4. Application to Perfect Fluids

Thus far, a specific distribution of gravitating matter was not assumed. If the matter tensor describing a perfect fluid with density ϱ, pressure p and normed velocity u^α of the streamlines ($u^2 = +1$) is chosen, i. e.

$$T^{\alpha\beta} = (p + \varrho)\, u^\alpha u^\beta - p\, g^{\alpha\beta} \tag{4.1}$$

then with the help of the field equations with cosmological constant Λ we may replace two components of $G_\beta{}^\alpha$ by the scalars ϱ and p.

It helps to distinguish two cases.

Case A:

$$G_1{}^0 \neq 0 .$$

Now, the following relations hold (for $p + \varrho \neq 0$):

$$\begin{aligned} &\varkappa p = \Lambda + G_2{}^2, \ \varkappa \varrho = -\Lambda - G_0{}^0 - G_1{}^1 + G_2{}^2 , \\ &u_0 u^0 = \frac{G_0{}^0 - G_2{}^2}{G_0{}^0 + G_1{}^1 - 2\, G_2{}^2}, \quad u_2 = u_3 = 0 , \\ &u_1 u^1 = \frac{G_1{}^1 - G_2{}^2}{G_0{}^0 + G_1{}^1 - 2\, G_2{}^2}, \\ &\Delta = (2\, G_2{}^2 + G_0{}^0 + G_1{}^1)^2 , \\ &G_1{}^0 G_0{}^1 = (G_0{}^0 - G_2{}^2)(G_1{}^1 - G_2{}^2) . \end{aligned} \tag{4.2}$$

The last of Eqs. (4.2) is discussed by Cahill and McVittie [9] and called consistency relation.

By choosing different signs of $\pm \Delta^{1/2}$ in Eq. (3.4) we obtain two possible second fundamental forms describing the local isometric embedding. The components of the first alternative are given by:

$$\begin{aligned} b_0{}^0 &= -\varepsilon_1\, \Omega_1{}^{-1/2} [\tfrac{1}{2} G_1{}^1 + \tfrac{1}{6} \varkappa \varrho + \tfrac{1}{6} \Lambda] , \\ b_1{}^1 &= -\varepsilon_1\, \Omega_1{}^{-1/2} [-\tfrac{1}{2} G_1{}^1 + \tfrac{1}{2} \varkappa p - \tfrac{1}{3} \varkappa \varrho - \tfrac{5}{6} \Lambda] , \\ b_1{}^0 &= \tfrac{1}{2} \varepsilon_1\, \Omega_1{}^{-1/2} G_1{}^0 , \\ b_2{}^2 &= b_3{}^3 = -e\, \varepsilon_1\, \Omega_1{}^{1/2} , \qquad \Omega_1 = -\tfrac{1}{3} e (\varkappa \varrho + \Lambda) . \end{aligned} \tag{4.3}$$

By choosing the other sign we derive the following components of the second fundamental form:

$$\begin{aligned} b_0{}^0 &= -\varepsilon_1\, \Omega_2{}^{-1/2} [\tfrac{1}{2} G_1{}^1 + \tfrac{1}{2} \varkappa p - \tfrac{1}{2} \Lambda] , \\ b_1{}^1 &= -\varepsilon_1\, \Omega_2{}^{-1/2} [-\tfrac{1}{2} G_1{}^1 + \varkappa p - \tfrac{1}{2} \varkappa \varrho - \tfrac{3}{2} \Lambda] , \\ b_1{}^0 &= \tfrac{1}{2} \varepsilon_1\, \Omega_2{}^{-1/2} G_1{}^0 , \\ b_2{}^2 &= b_3{}^3 = -e\, \varepsilon_1\, \Omega_2{}^{1/2} , \qquad \Omega_2 = e (\Lambda - \varkappa p) . \end{aligned} \tag{4.4}$$

In Eqs. (4.3), (4.4) $G_1{}^1$ and $G_1{}^0$ are given by the expressions of Eq. (2.5). They may be expressed as well by p, ϱ, Λ and the velocity component u^0 or u^1.

Case B:

$$G_1{}^0 = 0 .$$

Here, the relations are obtained:

$$\begin{aligned} &\varkappa p = \Lambda + G_2{}^2 , \varkappa \varrho = -\Lambda - G_0{}^0 , \\ &u^0 = 1 , \quad u^1 = u^2 = u^3 = 0 , \\ &\Delta = (3\, G_1{}^1 + G_0{}^0)^2 , \ G_2{}^2 = G_1{}^1 . \end{aligned} \tag{4.5}$$

Again, two possibilities for the second fundamental form arise. In the first case, $b_\beta{}^\alpha$ may be expressed completely by ϱ, p and Λ, i. e.:

$$\begin{aligned} b_0{}^0 &= -\varepsilon_1 P_1{}^{-1/2} [\tfrac{1}{2} \varkappa p + \tfrac{1}{6} \varkappa \varrho - \tfrac{1}{3} \Lambda] , \\ b_1{}^1 &= b_2{}^2 = b_3{}^3 = -e\, \varepsilon_1 P_1{}^{1/2} , \\ b_1{}^0 &= 0 , \quad P_1 = -\tfrac{1}{3} e (\varkappa \varrho + \Lambda) . \end{aligned} \tag{4.6}$$

The other possibility is:

$$\begin{aligned} -b_0{}^0 &= b_2{}^2 = b_3{}^3 = -e\, \varepsilon_1 P_2{}^{1/2} , \\ b_1{}^1 &= \varepsilon_1 P_2{}^{-1/2} [\tfrac{1}{2} \varkappa \varrho - \tfrac{1}{2} \varkappa p + \Lambda] , \\ b_1{}^0 &= 0 , \quad P_2 = e (\Lambda - \varkappa p) . \end{aligned} \tag{4.7}$$

Case B contains all static spherically symmetric metrics. We shall discuss it a bit further. Assume curvature coordinates, i. e. $\mu = \log r$, and put $e^{2\nu} = V^2(r)$, $e^{2\lambda} = A^2(r)$. Then, Eq. (3.2) reduces to

$$(A^{-2} - 1)\,) V'' + V' A' A^{-1} = 0 \tag{4.8}$$

which may be integrated to give three subcases.

a) $V' = 0$, A arbitrary

From $G_1{}^0 G_0{}^1 = (G_0{}^0 - G_2{}^2)(G_1{}^1 - G_2{}^2)$ holding for both cases A and B considered above, we obtain as the only possibilities

$$A^2 = \beta r^2 (1 + \beta r^2)^{-1}, \quad \beta \text{ a constant}, \tag{4.9 a}$$

$$A^2 = \alpha (\alpha - r^2)^{-1}, \quad \alpha \text{ a constant}. \tag{4.9 b}$$

If the metric is to correspond to a perfect fluid, only (4.9 b) can be kept. From Eqs. (2.4) and (3.3) we derive for (4.9 b)

$$b_0{}^0 = b_1{}^0 = 0 , \ b_1{}^1 = b_2{}^2 = b_3{}^3 = -e\, \varepsilon_1 (-e\, \alpha)^{-1/2} .$$

Thus, rank $b_\beta{}^\alpha = 3$ and the Codazzi equations (1.3b) ought to be checked separately. This gravitational field is equivalent, for $\Sigma = \sin x^2$, to the Einstein cosmological model for which an explicit embedding into E_5 is given by Rosen [11].

b) $A^2 = 1$, V arbitrary

Corresponding calculations lead to the expressions:

$$V = \alpha_1 \log r + \alpha_2 , \qquad (4.10\,a)$$

$$V = \tfrac{1}{2} \beta_1 r^2 + \beta_2 \ (\alpha_1, \alpha_2, \beta_1, \beta_2 \text{ constants}) . \qquad (4.10\,b)$$

In both cases we find $R_{23}{}^{23} = 0$ and thus cannot apply the formalism developed in Section 3. The metrics (4.10 a, b) are special cases of

$$ds^2 = c(t, r)\, dt^2 - dr^2 - r^2\, d\omega^2$$

studied in detail by Matsumoto and Kitamura [10].

c) $A^2 \neq 1$

If $A^2 \neq 1$, Eq. (4.8) may be integrated to give either

$$V' = \sigma (1 - A^2)^{1/2} \qquad (4.11\,a)$$

or

$$V' = \sigma (A^2 - 1)^{1/2} \ (\sigma \text{ constant}) . \qquad (4.11\,b)$$

For $\Sigma = \sin x^2$, these are the solutions discussed in detail by Kohler and Chao [12]. In fact, by putting $\Lambda = 0$, Eqs. (4.6), (4.7) coincide with corresponding expressions of Kohler and Chao. One of the solutions is equivalent to the interior Schwarzschild field, the other may be ruled out on physical grounds (It leads to $\varrho - 3p < 0$).

In conclusion two remarks seem to be in place:

1. It is a straightforward matter to apply these considerations to metrics with groups $G_3(2, t)$, i. e. whose group orbits are time-like (cf. Reference [3]).
2. Metrics with $G_3(2, s)$ and class 2 are no longer intrinsically (or energetically) rigid. Nevertheless, the procedure followed above can be used in order to express, by the Gauss equation (1.3 a), one of the two second fundamental forms by the other and the Einstein tensor.

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